Ergodic Theory - Week 12

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1 Entropy

In all the following exercises, we assume that we have a Borel probability space $(X, \mathcal{B}, \mu)$ and ξ, η will be partitions of $\mathcal{B}$.

P1. Prove the following properties of entropy of partitions:

- **(a)** If ξ is a finite partition with r atoms, then $H(\xi) \leq \log r$, with equality if and only if $\mu(A) = 1/r$ for any $A \in \xi$.
- **(b)** $H(\xi \vee \eta) = H(\eta) + H(\xi \mid \eta)$.
- **(c)** $H(\xi) \geq H(\xi \mid \eta)$.
- **(d)** Two partitions ξ and η are independent if and only if $H(\xi \mid \eta) = H(\xi)$.

Hint: In several places, you will need to use Jensen's inequality.

P2. Prove that $d(\xi, \eta) = H(\xi \mid \eta) + H(\eta \mid \xi)$ defines a metric in the space of finite partitions (up to sets of measure 0).

P3. Prove that for every $\epsilon > 0$ there is $\delta > 0$ such that $\xi = \{A_1, A_2, \ldots, A_r\}$ and $\eta = \{B_1, B_2, \ldots, B_r\}$ are two finite partitions with $\sum_{i=1}^r \mu(A_i \Delta B_i) < \delta$, then $d(\xi, \eta) < \epsilon$, where d is the metric defined in Problem 2.
Hint: Consider the partition $\gamma = \{A_i \cap B_j\}_{i \neq j} \cup \{\bigcup_{i=1}^r A_i \cap B_i\}$ and show that $H(\xi \mid \eta) \leq H(\gamma) < \varepsilon/2$ for δ small enough.